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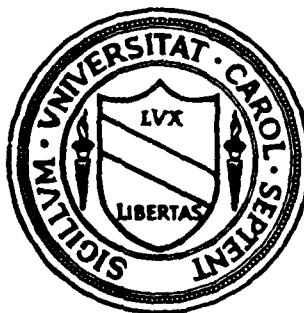
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ON A WIDE RANGE EXCLUSION PROCESS IN RANDOM MEDIUM WITH
LOCAL JUMP INTENSITY

by
E. Platen

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ON A WIDE RANGE EXCLUSION PROCESS IN RANDOM MEDIUM WITH
LOCAL JUMP INTENSITY

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Summary:

The paper investigates the macroscopic nonequilibrium dynamics of a wide range exclusion process in random medium. Based on a law of large numbers and the specific properties of the exclusion dynamics it is shown under suitable assumptions that the particle concentration follows a nonlinear evolution equation.

Key words: exclusion process
interacting particle system
nonequilibrium dynamics
nonlinear evolution equation

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1. Introduction

The exclusion process was introduced by Spitzer in /10/. Comprehensive treatments of this interacting particle system are given in /7/ and /1/.

In the exclusion model the particles attempt to move independently according to a Markov kernel on a given countable set of sites. But any jump which would take a particle to an already occupied site is suppressed. That means there is always at most one particle per site.

This paper is a continuation of /8/ where we considered the so called wide range birth and death exclusion process in random medium. We proved in /8/ a law of large numbers for this measure valued process and derived a deterministic macroscopic equation describing the evolution of the occupation rate of the sites. This macroscopic equation holds under rather general conditions. It can be specified by the choice of the Markov kernel, this means by the given jump intensity of the particles.

Within this paper we will study the case of a nonsymmetric local jump intensity which allows only jumps into the neighborhood. For instance, such jump intensity is of interest in modelling of stochastic charge transport (see /9/).

In the following we will show under suitable assumptions that the asymptotics with respect to vanishing mean jump

size yields a nonlinear second order partial differential equation characterizing the evolution of the limiting particle concentration. This equation represents a continuity equation, where physical quantities as current density vector, conductivity, static potential and chemical potential can be easily identified. It seems that this continuity equation models important transport processes in physics, chemistry, biology, electronics, social sciences and other fields.

The first part of the paper generalizes the birth and death exclusion process introduced in /8/. It formulates a law of large numbers for the case that the mean number of sites per unit volume tends to infinity. The second part derives the evolution equation for the limiting particle concentration.

2. Exclusion process

2.1. Random medium

In the following we will introduce a generalization of the wide range birth and death exclusion process considered in /8/.

For unexplained notations and definitions we refer to /4/ or /5/.

Let (Ω, \mathcal{F}, P) denote the basic complete probability space.

$\underline{F} = (F_t)_{t \geq 0}$ is an increasing right-continuous family of

complete sub- σ -fields of \underline{F} .

Further, $\underline{B}(E)$ represents the Borel- σ -algebra of a topological space E .

The sites are located in a closed bounded domain $\bar{Q} \subset \mathbb{R}^d$, $d \in \{1, 2, \dots\}$.

We introduce a finite σ -additive measure Λ on $\underline{B}(\bar{Q})$ which is called intensity measure of sites.

$(N_n)_{n>1}$ denotes a sequence of \underline{F}_0 -measurable simple counting measures on $\underline{B}(\bar{Q})$, the so called counting measures of sites.

The parameter

$$(2.1.1) \quad n = E N_n(\bar{Q}) \left(\int_{\bar{Q}} dq \right)^{-1}$$

can be interpreted as the mean number of sites per unit volume.

For $K \in (0, \infty)$ we denote by \underline{C}_K the set of bounded Lipschitz-continuous functions $f|_{\bar{Q}} \rightarrow [-K, K]$ with

$$(2.1.2) \quad |f(u) - f(q)| \leq K|q - u|$$

for all $u, q \in \bar{Q}$, using the usual Euclidean norm. Now we assume that for each $K \in (0, \infty)$ it holds

$$(2.1.3) \quad \lim_{n \rightarrow \infty} E \sup_{f \in \underline{C}_K} \left(\int_{\bar{Q}} f(q) \left(\frac{1}{n} N_n - \Lambda \right) (dq) \right)^2 = 0,$$

and for all $n > 1$ we have

$$(2.1.4) \quad E\left(\frac{1}{n} N_n(\bar{Q})\right)^4 = K_0 < \infty.$$

The counting measure N_n represents the random medium within our microscopic stochastic model. The above conditions are satisfied for a wide class of regular lattices and other point processes including the Poisson point process (see /8/).

2.2. Markovian jump mechanism

We denote by $L_{n,t}$ the counting measure of particles at time $t \geq 0$.

Further, we introduce an \underline{F} -adapted cadlag Poisson counting measure μ_n on $\underline{B}([0, \infty)) \otimes \underline{B}(\bar{Q}) \otimes \underline{B}(\bar{Q})$ which is characterized by its dual predictable projection which is here its intensity measure

$$(2.2.1) \quad \nu_n(dr, du, dq) = n^{-1} w_t(u, q) N_n(du) N_n(dq) dt.$$

We assume that the jump rate $w_t(u, q)$ is nonnegative and Lipschitz-continuous with respect to u and q uniformly with respect to t .

The counting measure μ_n generates the so called possible jumps of particles. But only jumps from occupied into vacant sites will take place. Therefore, we will have at most one particle at each site.

2.3. Birth and death of particles

We include in our microscopic stochastic model also the

effects of birth and death of particles.

The possible birth or death, resp., of particles is generated by the \underline{F} -adapted cadlag Poisson counting measures $\bar{\mu}_n$ and $\underline{\mu}_n$, resp., which are defined on $\underline{B}([0, \infty)) \otimes \underline{B}(\bar{Q})$ and characterized by their dual predictable projections

$$(2.3.1) \quad \bar{\gamma}_n(dt, du) = \bar{w}_t(u) N_n(du) dt$$

and

$$(2.3.2) \quad \underline{\gamma}_n(dt, du) = \underline{w}_t(u) N_n(du) dt$$

resp. The birth rate $\bar{w}_t(u)$ and death rate $\underline{w}_t(u)$ are assumed to be nonnegative bounded and Lipschitz-continuous with respect to $u \in \bar{Q}$ uniformly with respect to t .

Furthermore, we suppose that μ_n , $\bar{\mu}_n$ and $\underline{\mu}_n$ are independent. A birth (death, resp.) at u at time t will take place only if u is vacant (occupied, resp.) at this time. So also in the case of a birth or death it is ensured that we have at most one particle at each site.

2.4. Initial condition

The function $\gamma|Q \rightarrow [0,1]$ denotes the initial occupation rate of the sites. We assume that at time $t=0$ at most one particle is at each site, $L_{n,0}$ is \underline{F}_0 -measurable and for all $K \in (0, \infty)$ it holds

$$(2.4.1) \quad \lim_{n \rightarrow \infty} E \sup_{f \in \mathcal{C}_K} \left(\int_{\bar{Q}} f(q) \left(\frac{1}{n} L_{n,0} - \gamma(q) \wedge 1 \right) (dq) \right)^2 = 0.$$

2.5. Stochastic equation

Let δ_u denote the Dirac measure at $u \in \bar{Q}$ and $L_{n,t-}$ the left hand limit of the measure of particles at time $t > 0$.

Now we can define the measure valued exclusion process

$L_n = \{L_{n,t}\}_{t \geq 0}$ as unique solution of the following stochastic equation (see /8/)

$$(2.5.1) \quad L_{n,t} = L_{n,0} + \\ + \int_0^t \int_{\bar{Q}} \int_{\bar{Q}} (\delta_q - \delta_u) L_{n,s-}(\{u\}) (1 - L_{n,s-}(\{q\})) \mu_n(ds, du, dq) \\ + \int_0^t \int_{\bar{Q}} \delta_q (1 - L_{n,s-}(\{q\})) \bar{\mu}_n(ds, dq) \\ - \int_0^t \int_{\bar{Q}} \delta_u L_{n,s-}(\{u\}) \mu_n(ds, du),$$

which describes the evolution of L_n driven by μ_n , $\bar{\mu}_n$ and μ_n . One easily notes how the \mathcal{F} -adapted cadlag piecewise constant and Markovian measure valued process L_n remains at any time t with at most one particle at each site. Furthermore, the interaction between the particles caused by the exclusion mechanism is reflected by the logistic nonlinearity $L_n(1-L_n)$ within the second term of the right hand side of equation (2.5.1).

2.6. Occupation rate

Let \tilde{Q} denote the support of the intensity measure of sites Λ . Now, for $t \geq 0$ and $q \in \tilde{Q}$ we introduce the so called occupation rate $H(t, q)$ which we will later interpret as the probability that a site at $q \in \tilde{Q}$ is occupied at time t . We define the occupation rate H as the unique solution

(see /8/) of the following integro-differential equation

$$(2.6.1) \quad \frac{\partial}{\partial t} H(t, q) = (1 - H(t, q)) \left(\int_{\tilde{Q}} w_t(u, q) H(t, u) \Lambda(du) + \bar{w}_t(q) \right) - H(t, q) \left(\int_{\tilde{Q}} w_t(q, u) (1 - H(t, u)) \Lambda(du) + \underline{w}_t(q) \right)$$

for all $t > 0$, $q \in \tilde{Q}$, with initial condition:

$$(2.6.2) \quad H(0, q) = \varphi(q),$$

for all $q \in \tilde{Q}$. It can be shown as in /8/ that we have for all $t \geq 0$ and $q \in \tilde{Q}$:

$$(2.6.3) \quad H(t, q) \in [0, 1].$$

Equation (2.6.1) is a balance equation for the macroscopic evolution of the occupation rate and allows the following interpretation: The occupation rate changes its value in dependence on the occupation rate at other points. It increases at $q \in \tilde{Q}$ proportional to the non-occupation rate $(1 - H(t, q))$ and the sum of the birth rate $\bar{w}_t(q)$ together with the occupation rates $H(t, u)$ for the sites at $u \in \tilde{Q} \setminus \{q\}$ which are weighted by the intensity $w_t(u, q)$ for jumps from these sites into q . On the other hand $H(t, q)$ decreases proportional to its own actual value and the sum of the death rate $\underline{w}_t(q)$ together with the non-occupation rates $(1 - H(t, u))$ of the other sites at $u \in \tilde{Q} \setminus \{q\}$ which are weighted by the intensity $w_t(q, u)$ for the jumps from q into these sites.

We remark, that Groeger proved in /3/ that a unique solution exist for the balance equation (2.6.1) together with the Poisson equation describing the self consistent static potential.

2.7. Law of large numbers

Now, we can formulate a law of large numbers for the above introduced wide range exclusion process in random medium for the case that the mean number n of sites per unit volume tends to infinity, $n \rightarrow \infty$.

Theorem 2.7.1

Under the above assumptions it holds for fixed $T, K \in (0, \infty)$

$$(2.7.2) \lim_{n \rightarrow \infty} E \sup_{f \in \mathcal{C}_K} E \left(\sup_{0 \leq t \leq T} \left(\int_{\bar{Q}} f(q) \left(\frac{1}{n} L_{n,t} - H(t, q) / L \right) (dq) \right)^2 \middle| \mathcal{F}_0 \right) = 0.$$

The choice of the class of test functions \mathcal{C}_K and the positioning of the expectations are crucial for the proof of the theorem which uses semimartingale methods. It can be omitted because it would be almost the same as that which is given in /8/.

The above law of large numbers shows that the random functional

$$\int_{\bar{Q}} f(q) \frac{1}{n} L_{n,t} (dq)$$

converges in the mean square sense uniformly with respect

to the time $t \in [0, T]$ and all test functions $f \in \mathcal{C}_K$ for $n \rightarrow \infty$ to the deterministic functional

$$\int_0^T f(q) H(t, q) \lambda(dq).$$

The inner conditional expectation in (2.7.2) relates to the driving Poisson counting measures μ_n , $\bar{\mu}_n$ and $\underline{\mu}_n$. The remaining outer expectation averages the random medium and the random initial occupation of the sites.

3. Asymptotics of the macroscopic nonequilibrium dynamics

3.1. Specifications for the case of a local jump intensity

For simplicity let us choose the domain

$$(3.1.1) \quad Q = (0, l_1) \times \dots \times (0, l_d)$$

with

$$(3.1.2) \quad l_i \in (0, \infty)$$

for all $i \in \{1, \dots, d\}$.

∂Q is the boundary of Q and \bar{Q} denotes its closure.

Further, $\partial Q'$ is that part of ∂Q which does not contain corners or edges.

For $i, j \in \{0, 1, \dots\}$ we denote by \underline{C}^i the set of all i -times continuous differentiable functions on \bar{Q} , and $\underline{C}^{i,j}$ is the set of all functions on $[0, T] \times \bar{Q}$ which are i -times continuous differentiable with respect to the first and j -times with respect to the other coordinates, $T \in (0, \infty)$.

We denote by γ_q the outward unit normal of $\partial Q'$ at $q \in \partial Q'$, by $\text{div } a$ the divergence of a vector a , by $a \cdot e$ the usual scalar product between two vectors a and e and by $\text{grad } f$ the gradient of a function f on \mathbb{R}^d .

We assume that the intensity measure of sites is absolutely continuous and we write for all $q \in \bar{Q}$

$$(3.1.3) \quad \Lambda(dq) = \lambda(q) dq.$$

We suppose that there exists a version of the concentration of sites λ with the properties

$$(3.1.4) \quad \lambda(q) \geq K_1 > 0$$

for all $q \in \bar{Q}$, where

$$(3.1.5) \quad \lambda \in C^1.$$

Under this assumption we note that the support \tilde{Q} of Λ coincides with \bar{Q} . To formulate the local jump intensity we introduce the Lipschitz-continuous probability density

$$(3.1.6) \quad p(q) = \exp\{-q\} \left(\int_{\mathbb{R}^d} \exp\{-|u|\} du \right)^{-1}$$

for all $q \in \mathbb{R}^d$.

We remark that the proposed approach would also work for other probability densities if they show moment properties as those listed in Section 4.1.

For each value of a parameter $\alpha \in (0,1)$ we specify the local jump intensity for all $t \geq 0$ and $u, q \in \bar{Q}$ by the expression

$$(3.1.7) \quad w_t(u, q) = \pi(t, q)^{\frac{1}{2}} \pi(t, u)^{-\frac{1}{2}} p\left(\frac{2}{\alpha}(u-q)\right) \alpha^{-(d+2)}$$

with

$$(3.1.8) \quad \pi(t, q) = \exp\{-\bar{\Phi}(t, q)\},$$

where $\bar{\Phi}(t, q)$ is the so called static potential and we assume

$$(3.1.9) \quad \bar{\Phi} \in C^{1,1}.$$

We note that for smaller α the jump intensity w_t is more localized.

To interpret the jump intensity w_t we remark that one can show by the use of the properties of the probability density p listed in Section 4.1 that we have for $q \in Q$ the asymptotic drift vector

$$\lim_{\alpha \rightarrow 0} \int_Q (u-q) w_t(q, u) du = -b \operatorname{grad} \bar{\Phi}(t, q)$$

and the asymptotic variation

$$\lim_{\alpha \rightarrow 0} \int_Q (u-q)^2 w_t(q, u) du = bd,$$

where b is a positive constant depending on d . For instance we have for $d=1$ the value $b=1/2$ and for $d=3$ the value $b=7$.

Now, the occupation rate depends on the parameter $\alpha \in (0, 1)$ and we use in the following also the notation

$$(3.1.6) \quad H_{\alpha}(t,q) = H(t,q)$$

for all $t \in [0, T]$ and $q \in \bar{Q}$.

3.2. An integral equation

In the following we characterize the limiting dynamics of the occupation rate H_{α} for $\alpha \rightarrow 0$. For this purpose we assume that H_{α} converges pointwise to a function

$\bar{H} : [0, T] \times \bar{Q} \rightarrow [0, 1]$ such that

$$(3.2.1) \quad \lim_{\alpha \rightarrow 0} (H_{\alpha}(t,q) - \bar{H}(t,q)) = 0$$

for all $t \in [0, T]$ and $q \in \bar{Q}$.

Further, we assume that the time derivative of the occupation rate is uniformly bounded for all $\alpha \in (0, 1)$, $t \in [0, T]$ and $q \in \bar{Q}$ with

$$(3.2.2) \quad \left| \frac{\partial}{\partial t} H_{\alpha}(t,q) \right| \leq K_2.$$

Finally, let us suppose that there exists a constant

$$(3.2.3) \quad \alpha \in (0, 1/2)$$

such that for all $\alpha \in (0, 1)$, $q \in \bar{Q}$ and $t \in [0, T]$ we have

$$(3.2.4) \quad H_{\alpha}(t,q) \in [\alpha, 1-\alpha].$$

That means the occupation rate is never 0 or 1.

Let us denote by \tilde{C}^2 the set of functions $f \in C^2$ with

$$(3.2.5) \quad \frac{\partial}{\partial \nu_f} f(q) = 0$$

for all $q \in \partial Q'$.

Now, we are able to characterize \bar{H} as solution of an interesting integral equation.

Theorem 3.2.6

The limit \bar{H} of the occupation rate H_α for $\alpha \rightarrow 0$ satisfies under the above assumptions for all $f \in \tilde{C}^2$ and $t \in [0, T]$ the integral equation

$$\begin{aligned} (3.2.7) \quad & \int_{\bar{Q}} f(q) (\bar{H}(t, q) - \bar{H}(0, q)) \lambda(q) dq \\ &= \int_0^t \int_{\bar{Q}} \left\{ \frac{b}{2} \bar{H}(s, q) [\operatorname{div}(\lambda^2(q) \operatorname{grad} f(q)) \right. \\ & \quad - \lambda^2(q) (1 - \bar{H}(s, q)) \operatorname{grad} f(q) \operatorname{grad} \bar{\phi}(s, q)] \\ & \quad \left. + f(q) [(1 - \bar{H}(s, q)) \bar{w}_s(q) - \bar{H}(s, q) \underline{w}_s(q)] \lambda(q) \right\} dq ds. \end{aligned}$$

The proof of this theorem is given in Section 4.3.

Equation (3.2.7) gives a characterization of the limit \bar{H} which avoids smoothness assumptions on \bar{H} . Therefore one can say that (3.2.7) gives a rather weak description of \bar{H} . Under sufficient smoothness assumptions we will show within the next section that \bar{H} is the solution of a corresponding nonlinear partial differential equation which allows a direct interpretation of the dynamics

already described by the equation (3.2.7).

3.3. Asymptotic occupation rate

We assume the initial condition

$$(3.3.1) \quad \bar{H}(0, \cdot) \in \underline{C}^2$$

with

$$(3.3.2) \quad \frac{\partial}{\partial y_q} \bar{H}(0, q) + \bar{H}(0, q) (1 - \bar{H}(0, q)) \frac{\partial}{\partial y_q} \bar{\Phi}(0, q) = 0.$$

Now, we are going to introduce a function R on $[0, T] \times \bar{Q}$ which we call asymptotic occupation rate. We suppose that there exists a function $R \in \underline{C}^{1,2}$ which is the unique solution of the nonlinear partial differential equation

$$(3.3.3) \quad \begin{aligned} \frac{\partial}{\partial t} R(t, q) = & \lambda^1(q) \frac{b}{2} \operatorname{div}(\lambda^2(q) (\operatorname{grad} R(t, q) \\ & + R(t, q) (1 - R(t, q)) \operatorname{grad} \bar{\Phi}(t, q))) \\ & + (1 - R(t, q)) \bar{w}_t(q) - R(t, q) \underline{w}_t(q) \end{aligned}$$

for all $t > 0$ and $q \in Q$ with reflecting boundary condition

$$(3.3.4) \quad \frac{\partial}{\partial y_q} R(t, q) + R(t, q) (1 - R(t, q)) \frac{\partial}{\partial y_q} \bar{\Phi}(t, q) = 0$$

for all $t \geq 0$ and $q \in \partial Q'$, and initial condition

$$(3.3.5) \quad R(0, q) = \bar{H}(0, q)$$

for all $q \in \bar{Q}$.

We note that (3.3.3) represents a generalization of Burger's equation.

The following theorem shows under sufficient smoothness assumptions on the limit \bar{H} of the occupation rate H_α for $\alpha \rightarrow 0$, that \bar{H} coincides with the asymptotic occupation rate R .

Theorem 3.3.6

If we assume the property

$$(3.3.7) \quad \bar{H} \in \underline{C}^{7,2}$$

then we have for all $(t,q) \in [0,T] \times \bar{Q}$ the equivalence

$$(3.3.8) \quad H(t,q) = R(t,q).$$

The proof of this assertion is given in Section 4.4.

We remark that the smoothness assumption (3.3.7) on \bar{H} could be considerably weakened by an appropriate functional analytic formulation of the nonlinear partial differential equation (3.3.3) - (3.3.5). For instance such a weaker formulation could be based on methods described in /2/ or /6/. Here we have chosen rather strong smoothness assumptions on R and \bar{H} to derive the dynamics described by the equations (3.3.3) - (3.3.5) without technical difficulties in the formulation and prove of Theorem 3.3.6. It remains an interesting problem to derive these equations under much weaker assumptions.

Finally, we remark that the result could be generalized

to the case of a nonisotropic probability density p and a general regular domain \bar{Q} .

3.4. Continuity equation

For better interpretation of the asymptotic occupation rate R together with other physical quantities as particle concentration and current density vector we rewrite the equation (3.3.3) in the form of a continuity equation.

Let us introduce for all $t \geq 0$ and $q \in \bar{Q}$ the particle concentration

$$(3.4.1) \quad \rho(t, q) = R(t, q) \lambda(q)$$

and the vector function

$$(3.4.2) \quad j(t, q) = -\frac{h}{2} \lambda^2(q) (\text{grad } R(t, q) + R(t, q) (1 - R(t, q)) \text{ grad } \bar{\phi}(t, q))$$

which we will call current density vector.

Then it follows from (3.3.3) the continuity equation

$$(3.4.3) \quad \frac{\partial}{\partial t} \rho(t, q) = -\text{div}(j(t, q)) + (\lambda(q) - \rho(t, q)) \bar{w}_t(q) - \rho(t, q) \underline{w}_t(q)$$

for all $t > 0$ and $q \in Q$, with reflecting boundary condition

$$(3.4.4) \quad j(t, q) \cdot \bar{\nu}_q = 0$$

for all $t \geq 0$ and $q \in \partial Q'$, and initial condition

$$(3.4.5) \quad \rho(0, q) = \bar{H}(0, q) \lambda(q)$$

for all $q \in \bar{Q}$.

The above continuity equation relates the time derivative of the particle concentration with the current density vector.

We can also write the current density vector in the form

$$(3.4.6) \quad j(t, q) = -\frac{b}{2} \lambda^2(q) R(t, q) (1 - R(t, q)) \text{grad} (\mathcal{F}(t, q) + \Phi(t, q)),$$

for all $t \geq 0$, $q \in \bar{Q}$, where

$$(3.4.7) \quad \mathcal{F}(t, q) = \ln(R(t, q) (1 - R(t, q))^{-1})$$

is the so called chemical potential at $t \geq 0$ and $q \in \bar{Q}$.

One notes that the current density vector shows into the opposite direction of the gradient of the sum of the chemical and static potential.

The length of the current density vector is proportional to the so called conductivity

$$(3.4.8) \quad \sigma(t, q) = \frac{1}{2} b \lambda^2(q) R(t, q) (1 - R(t, q))$$

at $t \geq 0$ and $q \in \bar{Q}$, which contains a logistic nonlinearity with respect to the asymptotic occupation rate R . The

conductivity reaches its maximum at an asymptotic occupation rate with value $R=1/2$ which corresponds to the value of the chemical potential $\xi=0$. The logistic nonlinearity of the conductivity is caused by the interaction of the particles within the exclusion dynamics, where jumps to occupied sites are excluded. As important result of this nonlinearity it follows that the particle concentration is bounded by the concentration of sites and we have for all $t \geq 0$ and $q \in \bar{Q}$

$$(3.4.9) \quad 0 \leq \rho(t, q) \leq \lambda(q).$$

Finally, we note that there is no current through the boundary $\partial Q'$ what is natural in our model. It seems that the continuity equation (3.4.3)-(3.4.5) can be used to model important transport processes with birth and death effects in physics, chemistry, biology, social sciences and other fields.

4. Proofs

4.1 Properties of the probability density

For easier reference we list some properties of the probability density

$$p(q) = \exp\{-|q|\} \left(\int_{\mathbb{R}^d} \exp\{-|u|\} du \right)^{-1}$$

introduced in (3.1.4).

p is symmetric and we have for all $q \in \mathbb{R}^d$ that

$$(4.1.1) \quad p(q) = p(-q)$$

For all $i, k \in \{1, \dots, d\}$ one obtains by the use of the abbreviation

$$p = p\left(\frac{2}{\alpha}(u-q)\right) \alpha^{-(\alpha+2)}$$

the following moment properties

$$(4.1.2) \quad \lim_{\alpha \rightarrow 0} \int_{\bar{Q}} (u_i - q_i) p \, du = 0 \text{ for } q \in Q$$

$$(4.1.3) \quad \lim_{\alpha \rightarrow 0} \int_{\bar{Q}} (u_i - q_i) p \, du = d_i(q) \text{ for } q \in \partial Q,$$

$$(4.1.4) \quad \lim_{\alpha \rightarrow 0} \int_{\bar{Q}} (u_i - q_i)(u_k - q_k) p \, du = \begin{cases} b & \text{if } q \in Q \text{ and } i=k \\ 0 & \text{if } q \in Q \text{ and } i \neq k \\ c_{i,k}(q) & \text{if } q \in \partial Q \end{cases}$$

and

$$(4.1.5) \quad \lim_{\alpha \rightarrow 0} \int_{\bar{Q}} |u - q|^l p \, du = 0 \text{ for all } q \in \bar{Q} \text{ and } l \in \{3, 4, \dots\}$$

where b is a fixed bounded positive constant depending on the dimension d and it holds for d_i and $c_{i,k}$ the estimate

$$(4.1.6) \quad |c_{i,k}(q)| + |d_i(q)| \leq K$$

for all $q \in \partial Q$.

4.2. Some preparations

For all $\alpha \in (0, 1)$, $t \in [0, T]$ and $q \in \bar{Q}$ we define the function

$$(4.2.1) \quad A_\alpha(t, q) = H_\alpha(t, q) (1 - H_\alpha(t, q))^{-1} \pi(t, q)^{-1} \in [0, \infty]$$

and it follows with (3.2.4) that

$$(4.2.2) \quad H_\alpha(t, q) = A_\alpha(t, q) \pi(t, q) (1 + A_\alpha(t, q) \pi(t, q))^{-1} \in [\alpha, 1 - \alpha].$$

Now, we get from (4.2.1) with the estimate in (4.2.2) and the boundedness of π because of (3.1.8) and (3.1.9) also

an estimate for A_α with

$$(4.2.3) \quad A_\alpha(t, q) \in [\eta_1, \eta_2]$$

and

$$(4.2.4) \quad 0 < \eta_1 \leq \eta_2 < \infty$$

for all $\alpha \in (0, 1)$, $t \in [0, T]$ and $q \in \bar{Q}$.

With the notation (4.2.1) and the explicit form of the local jump intensity w_t in (3.1.7) we can write the integro-differential equation (2.6.1) in the form

$$(4.2.5) \quad \frac{\partial}{\partial t} H_\alpha(t, q) = \int_{\bar{Q}} (A_\alpha(t, u) - A_\alpha(t, q)) B \lambda(u) du + F_q$$

for all $\alpha \in (0, 1)$, $t \in (0, T]$ and $q \in \bar{Q}$ with the abbreviations

$$(4.2.6) \quad B = (1 - H_\alpha(t, q))(1 - H_\alpha(t, u)) \tau(t, q)^{\frac{1}{2}} \pi(t, u)^{\frac{1}{2}} p$$

and

$$(4.2.7) \quad F_q = (1 - H_\alpha(t, q)) \bar{w}_t(q) - H_\alpha(t, q) w_t(q).$$

To simplify our notation let us use also the following abbreviations

$$f_q = f(q), \lambda_q = \lambda(q); A_q = A_\alpha(t, q), H_q = H_\alpha(t, q), \bar{H}_q = \bar{H}(t, q), \\ \pi_q = \pi(t, q), \bar{\Phi}_q = \bar{\Phi}(t, q).$$

Further, we suppress the indication that we integrate over the set \bar{Q} .

Now, it follows from (4.2.5) and the symmetry of B for all

$\alpha \in (0,1)$, $t \in (0,T]$ and $f \in \subseteq_K$ that

$$(4.2.8) \quad \int_{\mathbb{R}^d} f \frac{\partial}{\partial t} H_q \lambda_q dq = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (A_u - A_q) B \lambda_u du \lambda_q dq + \int_{\mathbb{R}^d} f F_q \lambda_q dq \\ = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f_q - f_u) \frac{1}{2} (A_u - A_q) B \lambda_u \lambda_q du dq + \int_{\mathbb{R}^d} f F_q \lambda_q dq.$$

It holds the following continuity property for A_α .

Lemma 4.2.9

There exists a constant $K_3 \in (0, \infty)$ such that for each $t \in (0,T]$ and $\alpha \in (0,1)$ we have

$$(4.2.10) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} (A_q - A_u)^2 p du dq \leq K_3.$$

Proof

If we set in (4.2.8) $f_q = A_q$, then we obtain with (4.2.3), (3.2.2), (4.2.7), (3.2.4) and the boundedness of \bar{w}_t and \underline{w}_t that

$$(4.2.11) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} (A_q - A_u)^2 \frac{1}{2} B \lambda_u \lambda_q du dq = - \int_{\mathbb{R}^d} A_q \left(\frac{\partial}{\partial t} H_q + F_q \right) \lambda_q dq \leq K'.$$

Because of (4.2.6), (3.1.4), (3.2.4), (3.1.8) and (3.1.9) we can conclude

$$(4.2.12) \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} (A_q - A_u)^2 p du dq = K' (K_1^2 \alpha^2 K''')^{-1} \leq K_3. \quad \square$$

Using this result we obtain also an asymptotic continuity property for H :

Lemma 4.2.13

For all $t \in (0,T]$ we have

$$(4.2.14) \quad \lim_{\alpha \rightarrow 0} \iint (H_u - H_q)^2 |u-q|^2 p \, du dq = 0.$$

Proof

By the use of (3.2.4), (4.2.2), (3.1.8), Lemma (4.2.9) and (4.1.5) it follows for all $t \in (0, T]$ that

$$\begin{aligned} (4.2.15) \quad & \lim_{\alpha \rightarrow 0} \iint (H_u - H_q)^2 |u-q|^2 p \, du dq \\ &= \lim_{\alpha \rightarrow 0} \iint 2 |H_u - H_q| |u-q|^2 p \, du dq \\ &= \lim_{\alpha \rightarrow 0} \iint 2 (A_q |\pi_q - \pi_u| + \pi_u |A_q - A_u|) |u-q|^2 p \, du dq \\ &= \lim_{\alpha \rightarrow 0} \left\{ \iint K' |q-u|^3 p \, du dq + \iint K'' |A_q - A_u| |u-q|^2 p \, du dq \right\} \\ &= 0 + \lim_{\alpha \rightarrow 0} K'' \left(\int \left(\int |A_q - A_u| (u-q)^2 p \, du \right)^2 dq \right)^{\frac{1}{2}} \\ &= \lim_{\alpha \rightarrow 0} K''' \left(\int \left(\int |A_q - A_u|^2 p \, du \right) \left(\int (u-q)^4 p \, du \right) dq \right)^{\frac{1}{2}} \\ &= 0. \quad \square \end{aligned}$$

4.3. Proof of Theorem 3.2.6

We obtain from (4.2.8) with (4.2.1) for all $\alpha \in (0, 1)$, $t \in (0, T]$, and $f \in \tilde{C}^2$ by symmetry arguments

$$\begin{aligned} (4.3.1) \quad & \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial t} H_q - F_q \right) \lambda_q \, dq \\ &= \iint (f_q - f_u)^{\frac{1}{2}} \left\{ (1-H_q) H_u \pi_q^{\frac{1}{2}} \pi_u^{-\frac{1}{2}} \right. \\ &\quad \left. - (1-H_u) H_q \pi_u^{\frac{1}{2}} \pi_q^{-\frac{1}{2}} \right\} p \lambda_u \lambda_q \, du dq \\ &= \iint (f_q - f_u)^{\frac{1}{2}} \left\{ (H_u - H_q) + (1-H_q) H_u \left(\pi_q^{\frac{1}{2}} \pi_u^{-\frac{1}{2}} - 1 \right) \right. \\ &\quad \left. - (1-H_u) H_q \left(\pi_u^{\frac{1}{2}} \pi_q^{-\frac{1}{2}} - 1 \right) \right\} p \lambda_u \lambda_q \, du dq \end{aligned}$$

$$\begin{aligned}
 &= \iint (f_z - f_u) \left\{ -H_z + \frac{1}{2}(1-H_z)H_z \exp\left[\frac{1}{2}(\tilde{\Phi}_z - \tilde{\Phi}_u)\right] - 1 \right\} \\
 &\quad + \frac{1}{2}(1-H_z)H_z (1 - \exp\left[\frac{1}{2}(\tilde{\Phi}_z - \tilde{\Phi}_u)\right]) \} p \lambda_u \lambda_z \, du \, dz \\
 &= \iint (f_z - f_u) \left\{ -H_z + \frac{1}{4}(\tilde{\Phi}_u - \tilde{\Phi}_z) [(1-H_z)H_z \right. \\
 &\quad \left. + (1-H_u)H_z] + \tilde{R} \right\} p \lambda_u \lambda_z \, du \, dz \\
 &= \iint (f_z - f_u) \left\{ -H_z + \frac{1}{4}(\tilde{\Phi}_u - \tilde{\Phi}_z) [H_u(1-H_u) + H_z(1-H_z) \right. \\
 &\quad \left. + (H_u - H_z)^2] + \tilde{R} \right\} p \lambda_u \lambda_z \, du \, dz \\
 &= \iint (f_z - f_u) \left\{ -H_z + \frac{1}{2}(\tilde{\Phi}_u - \tilde{\Phi}_z) [H_z(1-H_z) + \frac{1}{2}(H_u - H_z)^2 \right. \\
 &\quad \left. + \tilde{R}] \lambda_z (\lambda_z + (\lambda_u - \lambda_z)) \right\} p \, du \, dz
 \end{aligned}$$

with

$$(4.3.2) \quad |\tilde{R}| \leq K_4 |\tilde{\Phi}_u - \tilde{\Phi}_z|^2 \leq K_5 |u - z|^2.$$

Now, we expand by the use of the Taylor formula the given functions f , $\tilde{\Phi}$ and λ , apply Lemma 4.2.13 and use the property (4.1.5) that higher moments than the second moment of p vanish asymptotically. Then it follows from the above result with assumption (3.2.1) and the other moment properties of p for all $t \in [0, T]$ and $f \in \underline{\tilde{C}}^2$ that

$$\begin{aligned}
 (4.3.3) \quad I &= \int_{\mathbb{R}^d} (H(t, q) - \bar{H}(0, q)) \lambda_q \, dq - \int_0^t \int_{\mathbb{R}^d} [(1 - \bar{H}(s, q)) \bar{w}_s(q) \\
 &\quad - \bar{H}(s, q) \underline{w}_s(q)] \lambda_q \, dq \, ds \\
 &= \lim_{\alpha \rightarrow 0} \left(\int_{\mathbb{R}^d} (H_\alpha(t, q) - H_\alpha(0, q)) \lambda_q \, dq - \int_0^t \int_{\mathbb{R}^d} f_q F_q \lambda_q \, dq \, ds \right)
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{\lambda \rightarrow 0^+} \left(\frac{1}{\lambda} \sum_{i=1}^d \frac{\partial}{\partial q_i} f_q(u_i - q_i) \right) \\
 &= \sum_{i,j=1}^d \frac{\partial^2}{\partial q_i \partial q_j} f_q(u_i - q_i)(u_j - q_j) \\
 &= \{ -H_q + \frac{1}{2} \left(\sum_{i=1}^d \frac{\partial}{\partial q_i} \bar{\phi}_q(u_i - q_i) \right) H_q (1 - H_q) \} \lambda_q \\
 &= \left(\lambda_q + \sum_{i=1}^d \frac{\partial}{\partial q_i} \lambda_q(u_i - q_i) \right) p \, du dq ds.
 \end{aligned}$$

We note that for $f \in \underline{\tilde{C}}^2$ we have for all $q \in \partial Q$ the boundary condition

$$\frac{\partial}{\partial \nu_q} f_q = 0.$$

Now, by the use of the moment properties of p : (4.1.2) - (4.1.5) and assumption (3.2.1) it follows from (4.3.3) for all $t \in [0, T]$ and $f \in \underline{\tilde{C}}^2$ the assertion

$$\begin{aligned}
 (4.3.4) \quad I &= \int_0^t \int b \left[-(\text{grad } f_q) \cdot \frac{1}{2} (\text{grad } \bar{\phi}_q) \bar{H}_q (1 - \bar{H}_q) \lambda_q^2 \right. \\
 &\quad \left. + (\text{grad } f_q) \cdot \bar{H}_q \lambda_q (\text{grad } \lambda_q) \right. \\
 &\quad \left. + \frac{1}{2} (\text{div grad } f_q) \bar{H}_q \lambda_q^2 \right] dq ds \\
 &= \int_0^t \int \bar{H}_q \left[\text{div}(\lambda_q^2 \text{grad } f_q) \right. \\
 &\quad \left. - \lambda_q^2 (1 - \bar{H}_q) \text{grad } f_q \cdot \text{grad } \bar{\phi}_q \right] dq ds. \quad \square
 \end{aligned}$$

4.4. Proof of Theorem 3.3.6

From Theorem 3.2.6 we obtain for all $t \in (0, T]$ and $f \in \underline{\tilde{C}}^2$ the equation

$$\begin{aligned}
 (4.4.1) \quad V &= \int_{\frac{1}{2}}^{\frac{1}{2}} \left(\frac{1}{2} \bar{H}_q - [(1-\bar{H}_q) \bar{H}_q(q) - \bar{H}_q w_q(q)] \right) \lambda_q dq \\
 &= \int_{\frac{1}{2}}^{\frac{1}{2}} \bar{H}_q \left[\operatorname{div}(\lambda_q^2 \operatorname{grad} f_q) \right. \\
 &\quad \left. - \lambda_q^2 (1-\bar{H}_q) (\operatorname{grad} f_q) \cdot (\operatorname{grad} \bar{H}_q) \right] dq \\
 &= \int_0^{1_d} \dots \int_0^{\frac{1}{2}} \sum_{i=1}^d \bar{H}_q \left[\frac{\partial}{\partial q_i} \left(\lambda_q^2 \frac{\partial}{\partial q_i} f_q \right) \right. \\
 &\quad \left. - \lambda_q^2 (1-\bar{H}_q) \left(\frac{\partial}{\partial q_i} f_q \right) \left(\frac{\partial}{\partial q_i} \bar{H}_q \right) \right] dq_1 \dots dq_d,
 \end{aligned}$$

and it follows by partial integration

$$\begin{aligned}
 V &= - \int_0^{1_d} \dots \int_0^{\frac{1}{2}} \sum_{i=1}^d \left(\frac{\partial}{\partial q_i} f_q \right) \lambda_q^2 \left(\frac{\partial}{\partial q_i} \bar{H}_q \right. \\
 &\quad \left. + \bar{H}_q (1-\bar{H}_q) \frac{\partial}{\partial q_i} \bar{H}_q \right) dq_1 \dots dq_d \\
 &\quad + \sum_{i=1}^d \int_0^{1_d} \dots \int_0^{1_{i-1}} \int_0^{1_{i+1}} \dots \int_0^{\frac{1}{2}} \left[\lambda_{\bar{v}_i}^2 \bar{H}_{\bar{v}_i} \frac{\partial}{\partial q_i} f_{\bar{v}_i} - \lambda_{\bar{v}_i}^2 \bar{H}_{\bar{v}_i} \frac{\partial}{\partial q_i} f_{\bar{v}_i} \right] \\
 &\quad dq_1 \dots dq_{i-1} dq_{i+1} \dots dq_d,
 \end{aligned}$$

where we used the abbreviations

$$\bar{v}_i = (q_1, \dots, q_{i-1}, 1, q_{i+1}, \dots, q_d)$$

and

$$v_i = (q_1, \dots, q_{i-1}, 0, q_{i+1}, \dots, q_d).$$

Because of (3.2.5) we can write for all $t \in (0, T]$ and $f \in \underline{\tilde{C}}^2$

$$(4.4.2) \quad V = \int_{\frac{1}{2}}^{\frac{1}{2}} \lambda_q^2 (\operatorname{grad} f_q) \cdot (\operatorname{grad} \bar{H}_q + \bar{H}_q (1-\bar{H}_q) \operatorname{grad} \bar{H}_q) dq.$$

Again by partial integration we get from (4.4.2) together with (4.4.1) for all $t \in (0, T]$ and $f \in \underline{\tilde{C}}^2$ the result

$$\begin{aligned}
 (4.4.3) \quad V &= \int_Q f_q \left(\frac{\partial}{\partial t} \bar{H}_q - [(1-\bar{H}_q) \bar{w}_t(q) - \bar{H}_q \underline{w}_t(q)] \right) \lambda_q dq \\
 &= \int_Q f_q \frac{b}{2} \operatorname{div} \left(\lambda_q^2 (\operatorname{grad} \bar{H}_q + \bar{H}_q (1-\bar{H}_q) \operatorname{grad} \bar{\phi}_q) \right) dq \\
 &\quad + \int_Q f_q \lambda_q^2 \left(\frac{\partial}{\partial \nu_q} \bar{H}_q + \bar{H}_q (1-\bar{H}_q) \frac{\partial}{\partial \nu_q} \bar{\phi}_q \right) dq.
 \end{aligned}$$

Because the above equation holds for all $f \in \tilde{C}^2$ and we assumed that \bar{H} is from $\tilde{C}^{1,2}$ it follows for \bar{H} with (3.1.4) and (3.3.2) the partial differential equation

$$\begin{aligned}
 (4.4.4) \quad &\left(\frac{\partial}{\partial t} \bar{H}_q - [(1-\bar{H}_q) \bar{w}_t(q) - \bar{H}_q \underline{w}_t(q)] \right) \lambda_q \\
 &= \frac{b}{2} \operatorname{div} \left(\lambda_q^2 (\operatorname{grad} \bar{H}_q + \bar{H}_q (1-\bar{H}_q) \operatorname{grad} \bar{\phi}_q) \right)
 \end{aligned}$$

for all $q \in Q$ and $t \in (0, T]$ with the boundary condition

$$(4.4.5) \quad \frac{\partial}{\partial \nu_q} \bar{H}_q + \bar{H}_q (1-\bar{H}_q) \frac{\partial}{\partial \nu_q} \bar{\phi}_q = 0$$

for all $q \in \partial Q$ and $t \in [0, T]$.

Now, we can conclude from (4.4.4) and (4.4.5) that \bar{H} is the unique solution of the equations (3.3.3)–(3.3.5) which means for all $t \in (0, T]$ and $q \in \bar{Q}$ that

$$(4.4.6) \quad \bar{H}(t, q) = R(t, q). \quad \square$$

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